6.1

а) Судећи по горњим и доњим индексима, можемо писати:

$$\overline{A}^{qts}_{p} = \ldots A_j^{klm}$$

b) $\overline{B}^{t}_{pq} = \ldots C^{m}_{jk}$ одн. B^{m}_{jk}

c) $\overline{C}^{p}_{ktn} = \ldots C^{j}_{qms}$, тј. треба бити $C(p,k,t,n) = \ldots C(j,q,m,s)$.

У зад. стоји $C(p,q,t,s) = \ldots C(j,k,m,n)$ па ова величина нема тензорску природу.

6.2

$$\overline{A}_t^{p} = \frac{\partial \overline{x}^p}{\partial x^q}\cdot\frac{\partial x^s}{\partial \overline{x}^t}\cdot \underset{\delta^t_q}{A^q_s} \Big/ \cdot \frac{\partial x^q}{\partial \overline{x}^p}$$

$$\frac{\partial x^q}{\partial \overline{x}^p}\overline{A}_t^{p} = \frac{\partial x^q}{\partial \overline{x}^p}\cdot\frac{\partial \overline{x}^p}{\partial x^q}\cdot\frac{\partial x^s}{\partial \overline{x}^t}\cdot A^q_s$$

како је: $\delta^k_l = \frac{\partial x^k}{\partial \overline{x}^i}\cdot\frac{\partial \overline{x}^i}{\partial x^l}$, то је: $\delta^q_q = \frac{\partial x^q}{\partial \overline{x}^p}\cdot\frac{\partial \overline{x}^p}{\partial x^q} = 1$

$$\frac{\partial x^q}{\partial \overline{x}^p}\overline{A}_t^{p} = \frac{\partial x^s}{\partial \overline{x}^t}\cdot A^q_s \Big/ \cdot \frac{\partial \overline{x}^t}{\partial x^s}$$

$$\frac{\partial \overline{x}^t}{\partial x^s}\cdot\frac{\partial x^q}{\partial \overline{x}^p}\overline{A}_t^{p} = A^q_s$$

... аријантних вектора и тензора.

6.3 Закон трансформ. контравар. вектора:

а) $\bar{A}^i = \dfrac{\partial \bar{x}^i}{\partial x^j} A^j$ $\quad i = 1,2,3;\ j = 1,2,3$

→ сумма по j!

$$\bar{A}^1 = \frac{\partial \bar{x}^1}{\partial x^1} A^1 + \frac{\partial \bar{x}^1}{\partial x^2} A^2 + \frac{\partial \bar{x}^1}{\partial x^3} A^3$$

$$\bar{A}^2 = \frac{\partial \bar{x}^2}{\partial x^1} A^1 + \frac{\partial \bar{x}^2}{\partial x^2} A^2 + \frac{\partial \bar{x}^2}{\partial x^3} A^3$$

$$\bar{A}^3 = \frac{\partial \bar{x}^3}{\partial x^1} A^1 + \frac{\partial \bar{x}^3}{\partial x^2} A^2 + \frac{\partial \bar{x}^3}{\partial x^3} A^3$$

$$\begin{pmatrix} \bar{A}^1 \\ \bar{A}^2 \\ \bar{A}^3 \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^1}{\partial x^3} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \frac{\partial \bar{x}^2}{\partial x^3} \\ \frac{\partial \bar{x}^3}{\partial x^1} & \frac{\partial \bar{x}^3}{\partial x^2} & \frac{\partial \bar{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} A^1 \\ A^2 \\ A^3 \end{pmatrix}$$

в) Ковариантн. вектора

$\bar{A}_i = \dfrac{\partial x^j}{\partial \bar{x}^i} \cdot A_j$ → сумма по j

$$\bar{A}_1 = \frac{\partial x^1}{\partial \bar{x}^1} \cdot A_1 + \frac{\partial x^2}{\partial \bar{x}^1} \cdot A_2 + \frac{\partial x^3}{\partial \bar{x}^1} \cdot A_3$$

$$\bar{A}_2 = \frac{\partial x^1}{\partial \bar{x}^2} \cdot A_1 + \frac{\partial x^2}{\partial \bar{x}^2} \cdot A_2 + \frac{\partial x^3}{\partial \bar{x}^2} \cdot A_3$$

$$\bar{A}_3 = \frac{\partial x^1}{\partial \bar{x}^3} \cdot A_1 + \frac{\partial x^2}{\partial \bar{x}^3} \cdot A_2 + \frac{\partial x^3}{\partial \bar{x}^2} \cdot A_3$$

$$\begin{pmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \bar{A}_3 \end{pmatrix} = \begin{pmatrix} \frac{\partial x^1}{\partial \bar{x}^1} & \frac{\partial x^2}{\partial \bar{x}^1} & \frac{\partial x^3}{\partial \bar{x}^1} \\ \frac{\partial x^1}{\partial \bar{x}^2} & \frac{\partial x^2}{\partial \bar{x}^2} & \frac{\partial x^3}{\partial \bar{x}^2} \\ \frac{\partial x^1}{\partial \bar{x}^3} & \frac{\partial x^2}{\partial \bar{x}^3} & \frac{\partial x^3}{\partial \bar{x}^3} \end{pmatrix} \begin{pmatrix} A_1 \\ A_2 \\ A_3 \end{pmatrix}$$

Закон трансформације контраваријантних тензора 6.24

$$\bar{A}^{ij} = \frac{\partial \bar{x}^i}{\partial x^k}\cdot\frac{\partial \bar{x}^j}{\partial x^l} A^{kl} \quad \rightarrow \text{сумирање по } k \text{ и } l$$

$$\begin{pmatrix} \bar{A}^{11} & \bar{A}^{12} & \bar{A}^{13} \\ \bar{A}^{21} & \bar{A}^{22} & \bar{A}^{23} \\ \bar{A}^{31} & \bar{A}^{32} & \bar{A}^{33} \end{pmatrix} = \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^1}{\partial x^3} \\ \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^2} & \frac{\partial \bar{x}^2}{\partial x^3} \\ \frac{\partial \bar{x}^3}{\partial x^1} & \frac{\partial \bar{x}^3}{\partial x^2} & \frac{\partial \bar{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} \frac{\partial \bar{x}^1}{\partial x^1} & \frac{\partial \bar{x}^2}{\partial x^1} & \frac{\partial \bar{x}^3}{\partial x^1} \\ \frac{\partial \bar{x}^1}{\partial x^2} & \frac{\partial \bar{x}^2}{\partial x^2} & \frac{\partial \bar{x}^3}{\partial x^2} \\ \frac{\partial \bar{x}^1}{\partial x^3} & \frac{\partial \bar{x}^2}{\partial x^3} & \frac{\partial \bar{x}^3}{\partial x^3} \end{pmatrix} \begin{pmatrix} A^{11} & A^{12} & A^{13} \\ A^{21} & A^{22} & A^{23} \\ A^{31} & A^{32} & A^{33} \end{pmatrix}$$

6.4 $A(p,q,r) = ?$

Ако је $A(p,q,r)\cdot B^{qs}_{r} = C^{s}_{p}$, тада је $A(p,q,r) = A^{r}_{pq}$ јер је

$A^{r}_{pq}\cdot B^{qs}_{r} = C^{s}_{p}$ → спољашњи производ тензора + контракција = унутрашњи производ тензора. По критеријуму количника ово је тензор и то трећег ранга → два пута коваријантан и једанпут контраваријантан.

(6.5) $\phi \to$ инваријанта и $\phi = \phi(x^1, x^2, \ldots, x^n)$; $\bar{\phi} = (\bar{x}^1, \bar{x}^2, \ldots, \bar{x}^n)$

$$\frac{\partial^2 \bar{\phi}}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial^2 \phi}{\partial \bar{x}^i \partial \bar{x}^j} = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \phi}{\partial \bar{x}^j}\right) = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \phi(x^k)}{\partial \bar{x}^j}\right) = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \phi}{\partial x^k} \cdot \frac{\partial x^k}{\partial \bar{x}^j}\right) =$$

$$= \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \phi}{\partial x^k}\right) \cdot \frac{\partial x^k}{\partial \bar{x}^j} + \frac{\partial \phi}{\partial x^k} \cdot \frac{\partial^2 x^k}{\partial \bar{x}^i \cdot \partial \bar{x}^j} =$$

$$= \frac{\partial}{\partial x^l}\left(\frac{\partial \phi}{\partial x^k}\right) \frac{\partial x^l}{\partial \bar{x}^i} \cdot \frac{\partial x^k}{\partial \bar{x}^j} + \frac{\partial \phi}{\partial x^k} \cdot \frac{\partial^2 x^k}{\partial \bar{x}^i \cdot \partial \bar{x}^j} =$$

$$= \frac{\partial x^l}{\partial \bar{x}^i} \cdot \frac{\partial x^k}{\partial \bar{x}^j} \cdot \frac{\partial^2 \phi}{\partial x^k \partial x^l} + \frac{\partial \phi}{\partial x^k} \cdot \frac{\partial^2 x^k}{\partial \bar{x}^i \cdot \partial \bar{x}^j}$$ → коваријантна величина није тензор због другог члана

Да није њега ово би било коваријантни тензор.

(6.6) $A_i \to$ коваријантни вектор, тј. $\bar{A}_i = \frac{\partial x^k}{\partial \bar{x}^i} A_k$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} = \frac{\partial}{\partial \bar{x}^j}\left(\frac{\partial x^k}{\partial \bar{x}^i} A_k\right) = \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} A_k + \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial A_k}{\partial \bar{x}^j} =$$

$$= \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} A_k + \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial A_k}{\partial x^l} \cdot \frac{\partial x^l}{\partial \bar{x}^j} =$$

$$= \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial A_k}{\partial x^l} + \frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} A_k$$ → није тензор због другог члана

јер је $\frac{\partial^2}{\partial \bar{x}^i \partial \bar{x}^j}\left(x^k A_k - x^l A_l\right)$ ↓ сумира се по k, сумира се по l, па је ово исто што $x^1 A_1 - x^1 A_1$ за свако

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \frac{\partial \bar{A}_j}{\partial \bar{x}^i} = ?$$

$$\frac{\partial \bar{A}_i}{\partial \bar{x}^j} - \frac{\partial \bar{A}_j}{\partial \bar{x}^i} = \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial A_k}{\partial x^l} + \cancel{\frac{\partial^2 x^k}{\partial \bar{x}^i \partial \bar{x}^j} A_k} - \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial A_l}{\partial x^k} - \cancel{\frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} A_l}$$

$$= \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial x^l}{\partial \bar{x}^j}\left(\frac{\partial A_k}{\partial x^l} - \frac{\partial A_l}{\partial x^k}\right)$$ → двоструки коваријантни тензор

$$\frac{\partial \bar{A}_j}{\partial \bar{x}^i} = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial x^l}{\partial \bar{x}^j} A_l\right) = \frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} A_l + \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial A_l}{\partial \bar{x}^i} = \frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} A_l + \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial A_l}{\partial x^k} \cdot \frac{\partial x^k}{\partial \bar{x}^i} =$$

$$= \frac{\partial^2 x^l}{\partial \bar{x}^i \partial \bar{x}^j} A_l + \frac{\partial x^l}{\partial \bar{x}^j} \cdot \frac{\partial x^k}{\partial \bar{x}^i} \cdot \frac{\partial A_l}{\partial x^k}$$ ✓

6.7 $\quad \bar{A}^i = \dfrac{\partial \bar{x}^i}{\partial x^j} A^j$

$$\frac{\partial \bar{A}^i}{\partial \bar{x}^i} = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \bar{x}^i}{\partial x^j} A^j\right) = \frac{\partial}{\partial \bar{x}^i}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) A^j + \frac{\partial \bar{x}^i}{\partial x^j} \cdot \frac{\partial A^j}{\partial \bar{x}^i} = \frac{\partial}{\partial x^j}\left(\frac{\partial \bar{x}^i}{\partial \bar{x}^i}\right) A^j +$$

$\delta^i_i = 3$

$+ \dfrac{\partial A^j}{\partial x^j}$ → скалар! (као што је дивергенција скалар)

6.9 $\quad B_i = \dfrac{\partial x^j}{\partial \bar{x}^i} B_j$; $\quad x^1 x^2,\ 2x^2 - (x^3)^2,\ x^1 x^3$; нове сферне коорд. $r \equiv \bar{x}^1$, $\theta \equiv \bar{x}^2$, $\varphi \equiv \bar{x}^3$

па је:
$$x^1 = \bar{x}^1 \cdot \sin \bar{x}^2 \cdot \cos \bar{x}^3$$
$$x^2 = \bar{x}^1 \sin \bar{x}^2 \cdot \sin \bar{x}^3$$
$$x^3 = \bar{x}^1 \cdot \cos \bar{x}^2$$

веза Декартових и сферних коорд.
$$x^1 = r \sin\theta \cos\varphi$$
$$x^2 = r \sin\theta \sin\varphi$$
$$x^3 = r\cos\theta$$

На основу резултата задатка 6.3 пишемо:

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} \dfrac{\partial x^1}{\partial \bar{x}^1} & \dfrac{\partial x^2}{\partial \bar{x}^1} & \dfrac{\partial x^3}{\partial \bar{x}^1} \\[2ex] \dfrac{\partial x^1}{\partial \bar{x}^2} & \dfrac{\partial x^2}{\partial \bar{x}^2} & \dfrac{\partial x^3}{\partial \bar{x}^2} \\[2ex] \dfrac{\partial x^1}{\partial \bar{x}^3} & \dfrac{\partial x^2}{\partial \bar{x}^3} & \dfrac{\partial x^3}{\partial \bar{x}^3} \end{pmatrix} \begin{pmatrix} B_1 \\ B_2 \\ B_3 \end{pmatrix}$$

Сада је:

$$\begin{pmatrix} \bar{B}_1 \\ \bar{B}_2 \\ \bar{B}_3 \end{pmatrix} = \begin{pmatrix} \sin \bar{x}^2 \cos \bar{x}^3 & \sin \bar{x}^2 \cdot \sin \bar{x}^3 & \cos \bar{x}^2 \\ \bar{x}^1 \cos \bar{x}^2 \cos \bar{x}^3 & \bar{x}^1 \cos \bar{x}^2 \sin \bar{x}^3 & -\bar{x}^1 \sin \bar{x}^2 \\ -\bar{x}^1 \sin \bar{x}^2 \sin \bar{x}^3 & \bar{x}^1 \sin \bar{x}^2 \cos \bar{x}^3 & 0 \end{pmatrix} \times \begin{pmatrix} B_1 = (\bar{x}^1)^2 \sin^2 \bar{x}^2 \cdot \sin \bar{x}^3 \cos \\ B_2 = 2\bar{x}^1 \sin \bar{x}^2 \sin \bar{x}^3 - (\bar{x}^1)^2 \cos \\ B_3 = (\bar{x}^1)^2 \sin \bar{x}^2 \cos \bar{x}^2 \cos \bar{x}^3 \end{pmatrix}$$

Након множења матрица добијено:

$$\bar{B}_1 = (\bar{x}^1)^2 \sin \bar{x}^2 \left(\sin^2 \bar{x}^2 \cdot \cos^2 \bar{x}^3 \sin \bar{x}^3 - \cos^2 \bar{x}^2 \sin \bar{x}^3 + \cos^2 \bar{x}^2 \cos \bar{x}^3\right) + 2\bar{x}^1 \sin^2 \bar{x}^2 \sin^2 \bar{x}^3$$

$$\bar{B}_2 = (\bar{x}^1)^3 \left(\sin^2 \bar{x}^2 \cos \bar{x}^2 \sin \bar{x}^3 \cos^2 \bar{x}^3 - \cos^3 \bar{x}^2 \sin \bar{x}^3 - \sin^2 \bar{x}^2 \cos \bar{x}^2 \cos \bar{x}^3\right) + 2(\bar{x}^1)^2 \sin \bar{x}^2 \cos \bar{x}^2 \sin^2$$

$$\bar{B}_3 = 2(\bar{x}^1)^2 \sin^2 \bar{x}^2 \sin \bar{x}^3 \cos \bar{x}^3 - (\bar{x}^1)^3 \left(\sin^3 \bar{x}^2 \sin^2 \bar{x}^3 \cos \bar{x}^3 + \sin \bar{x}^2 \cos^2 \bar{x}^2 \cos \bar{x}^3\right)$$

Ово су компоненте датог коваријантног вектора у сферним коорд.

6.10 $v^i = \frac{dx^i}{dt}$ → контраваријантни вектор, тј. треба доказати да важи:

$$\bar{v}^i = \frac{\partial \bar{x}^i}{\partial x^j} \cdot v^j \; ; \; x^i = x^i(t)$$

Како је: $\bar{v}^i = \frac{d\bar{x}^i}{dt} = \frac{\partial \bar{x}^i}{\partial x^j} \cdot \overset{v^j}{\frac{dx^j}{dt}} = \frac{\partial \bar{x}^i}{\partial x^j} \cdot v^j$ → контраварир. вектор!

$$= \frac{\partial}{\partial x^j}\left(\frac{d\bar{x}}{dt}\right) v^j$$

Убрзање: $\bar{a}^i = \frac{d\bar{v}^i}{dt} = \frac{d}{dt}\left(\frac{\partial \bar{x}^i}{\partial x^j} v^j\right) = \frac{d}{dt}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) v^j + \frac{\partial \bar{x}^i}{\partial x^j} \cdot \frac{dv^j}{dt} =$

$$= \frac{\partial}{\partial x^k}\left(\frac{\partial \bar{x}^i}{\partial x^j}\right) \cdot \frac{dx^k}{dt} v^j + \frac{\partial \bar{x}^i}{\partial x^j} a^j =$$

$$= \frac{\partial^2 \bar{x}^i}{\partial x^j \partial x^k} v^j v^k + \frac{\partial \bar{x}^i}{\partial x^j} a^j$$

→ ово није ни коваријантни ни (осим за случај лин. трансф.) контраваријантни вектор, па убрзање није вектор у смислу општег тензорског рачуна